2009-2010 Puzzle Contest #5

Solutions

Parents and Grandparents Puzzle Solutions:

1. THE FRACTIONAL CASE

WHAT NATURAL numbers N can be written as $\frac{1}{n_1}+\frac{2}{n_2}+\frac{3}{n_3}+\cdots+\frac{119}{n_{119}}$, where $n_1, n_2, n_3, \dots, n_{119}$ are natural numbers? **50 points**

Answer: All the natural numbers from $[\mathbf{1}; \mathbf{7140}]$

Solution: The biggest number which can be written as above is $N_{max} = 1+2+3+\cdots+119 = 7140$. It can be obtained for $n_1 = 1, n_2 = 1, n_3 = 1, \dots, n_{119} = 1$. The number 119 can be obtained for $n_1 = 1, n_2 = 2, n_3 = 3, \dots, n_{119} = 119$. *Let be* $k < 119$ – *some* number. Then $k = \underbrace{1+1+1+\cdots+1}_{(k-1)} + \underbrace{\frac{1}{n-k+1}+\frac{1}{n-k+1}+\frac{1}{n-k+1}+\cdots+\frac{1}{n-k+1}}_{(n-k+1)} = \frac{1}{1}+\frac{2}{2}+\cdots+\frac{k-1}{k-1}+\frac{k}{k(n-k+1)}+\frac{k+1}{(k+1)(n-k+1)}+\cdots+\frac{119}{119(n-k+1)}$. It means N can be any natural number less than 119. If $119 < k < 7140, then\ k = 119 + 118 + \cdots + (l+1) + \underbrace{1+1+1+\cdots+1}_{l}$ $for\ some\ natural\ l. So\ n_1 = 1, n_2 = 2, \dots, n_l = l,\ n_{l+1} = n_{l+2} = \cdots = n_{119} = 1$. It means N can be any natural number between 119 and 7140. **And finally $N \in [\mathbf{1}; \mathbf{7140}]$**

2. A CONVEX POLYGON...

ALL POSSIBLE DIAGONALS are drawn in some n-side convex polygon. There are no three among them having the same intersection point within the polygon. What is the relationship between n and a number of all possible intersection points of diagonals within the polygon? **30 points**

$\boldsymbol{Answer}: \boldsymbol{N = \frac{n(n-1)(n-2)(n-3)}{24}}$. $\boldsymbol{Here\ are\ N - the\ number\ of\ points,\quad n - the\ number\ of\ diagonals}$

Solution: Every intersection point can be formed by two intersected diagonals only. Four ends of those diagonals are vertices of n-side convex polygon and can form exactly one convex quadrilateral. Hence the number of intersection points is equal to the number of convex quadrilaterals which are formed by all possible vertices of the initial polygon. It is the number of combination of 4 out of n, e.g. $C_n^4 = \frac{\boldsymbol{n(n-1)(n-2)(n-3)}}{\boldsymbol{4!}}$. **Please note that the sentence** "There are no three among them having the same intersection point within the polygon" **is important for the solution(for example, check the regular hexagon)** .

3. "SERIOUS" DIVISION

EVERY LETTER of the Cryptogram $\frac{\overline{\boldsymbol{AHHAAH}}}{\overline{\boldsymbol{JOKE}}} = \overline{\boldsymbol{HA}}$ represents a non-zero decimal digit uniquely. What is $\overline{\boldsymbol{JOKE}}$? **20 points**

Answer: $\overline{JOKE}$=5169.

Solution: $\overline{JOKE}=\frac{\overline{AHHAAH}}{\overline{HA}} = 100 + \frac{\overline{AH}(10001)}{\overline{HA}} = 100 + \frac{\overline{AH}\cdot 73\cdot 137}{\overline{HA}}$. Then $\overline{HA} = 73$. Hence $\overline{JOKE} = \frac{377337}{73} = 5169$